Towards Trace Metrics via Functor Lifting

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CALCO 2015 – June 26, 2015

My Talk Today

Introduction

Behavioral Equivalence vs Behavioral Metric

Behavioral Equivalence

Two systems behave exactly the same.

(bisimilarity, trace/language equivalence, failure equivalence ...)

Behavioral (Pseudo-)Metrics

Two systems behave similarly, i.e. they have a small distance.

(bisimilarity pseudometric, trace/language pseudometric ...)

Our Central Contributions

- FSTTCS ’14: Behavioral pseudometrics for coalgebras of $\mathbf{Set}$-functors.
- This talk: Trace pseudometrics (using Eilenberg-Moore).

Examples of Trace Distances

- NFA $L_x, L_y \colon A^* \to 2$

$$d_{2^{A^*}}(L_x, L_y) = c^{\min\{n \in \mathbb{N}_0 \mid \exists w \in A^n . L_x(w) \neq L_y(w)\}}$$

  where $c \in (0, 1)$.

- Probabilistic Moore automata $p_x, p_y \colon A^* \to [0, 1]$

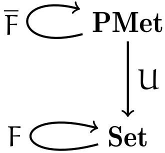
$$d_{[0,1]^{A^*}}(p_x, p_y) = c_1 \cdot \sum_{w \in A^*} \left(\frac{c_2}{|A|}\right)^{|w|} |p_x(w) - p_y(w)| \ .$$

  where $c_1, c_2 \in (0, 1]$, $c_1 + c_2 \leqslant 1$.

Lifting Functors [FSTTCS ’14]

Lifting of a Functor

[FSTTCS ’14]



Definition: Lifting of a Functor (to $\mathbf{PMet}$)

$\overline{F}$ is a lifting of F if the diagram commutes.

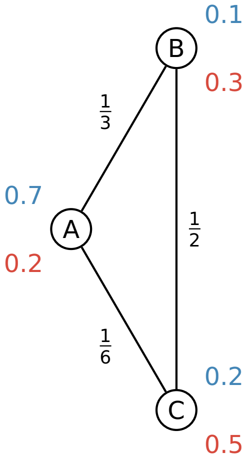
In Plain Terms: Our Task

Given: $F\colon \mathbf{Set} \to \mathbf{Set}$ and a pseudometric space (X, d)
Define: pseudometric d^F on FX

Pseudometrics on Probability Distributions

Example Based on Optimal Transportation [Villani, 2009]

$X = \{A, B, C\}$, $\quad d_X : X \times X \to [0, 1]$, $\quad \mu, \nu : X \to [0, 1]$



Transportation Problem

- A, B, C: producers/consumers
- supply: μ, demand: ν (in %)
- transporting along d costs d
- minimize total cost of transport

Optimal Solution

- first use local production (cost 0)
- transport 0.2 from A to B: $\frac{1}{3} \cdot 0.2 = \frac{1}{15}$
- transport 0.3 from A to C: $\frac{1}{6} \cdot 0.3 = \frac{1}{20}$

$$d(\mu, \nu) = \frac{1}{15} + \frac{1}{20} = \frac{7}{60}$$

$$\mu, \nu \in \mathcal{D}(X), \quad t \in \mathcal{D}(X \times X), \quad t \colon \underbrace{X}_{\text{from}} \times \underbrace{X}_{\text{to}} \to \underbrace{[0,1]}_{\text{percentage}}$$

supply is used up

$$\forall x \in X. \sum_{y \in X} t(x, y) = \mu(x)$$

$$\iff \quad \mathcal{D}\pi_1(t) = \mu$$

demand is satisfied

$$\forall y \in X. \sum_{x \in X} t(x, y) = \nu(x)$$

$$\iff \quad \mathcal{D}\pi_2(t) = \nu$$

Definition: Coupling

$F \colon \mathbf{Set} \to \mathbf{Set}$, $t_1, t_2 \in FX$. The set of couplings of t_1 and t_2 is

$$\Gamma_F(t_1, t_2) := \left\{ t \in F(X \times X) \;\middle|\; \begin{matrix} F\pi_1(t) = t_1 \\ F\pi_2(t) = t_2 \end{matrix} \right\}.$$

$$\mathsf{costs}_d(t) = \sum_{(x,y)\in X\times X} t(x,y)\cdot d(x,y)$$

$$d\colon X\times X \to [0,1] \qquad \leadsto \qquad \mathsf{costs}_d\colon \mathcal{D}(X\times X) \to [0,1]$$

Definition: Evaluation Functor & Evaluation Function

$$\begin{array}{ccc} \mathbf{Set}/[0,\top] & \xrightarrow{\widetilde{F}} & \mathbf{Set}/[0,\top] \\ X \xrightarrow{f} [0,\top] & \mapsto & FX \xrightarrow{Ff} \underbrace{F[0,\top] \xrightarrow{ev_F} [0,\top]}_{\text{evaluation function}}. \end{array}$$

Putting It Together: The Wasserstein Lifting

[FSTTCS '14]

Definition: Wasserstein Distance

For $t_1, t_2 \in FX$ define

$$d^{\downarrow F}(t_1, t_2) = \inf \left\{ \widetilde{F}d(t) \;\middle|\; t \in \Gamma_F(t_1, t_2) \right\} .$$

Proposition: Wasserstein Pseudometric

$d^{\downarrow F}$ is a pseudometric if

- F preserves weak pullbacks
- ev_F satisfies some constraints (well-behaved)

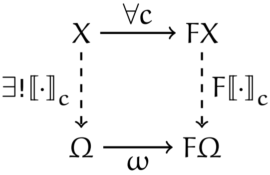
Behavioral (Pseudo-)Metrics [FSTTCS ’14]

Final Coalgebra and Behavior

The Essential Basics in a Nutshell (i.e. on one Slide)

Definition: Final Coalgebra

$\omega\colon \Omega \to F\Omega$ is final, if for all $c\colon X \to FX$ there is a unique coalgebra homomorphism $[\![\cdot]\!]_c\colon X \to \Omega$.



Definition: Behavioral Equivalence

$$\forall x, y \in X: \qquad x \sim_c y \quad \Longleftrightarrow \quad [\![x]\!]_c = [\![y]\!]_c$$

Definition: Behavioral Pseudometric [FSTTCS '14]

$$\forall x, y \in X: \qquad d_c(x, y) \quad = \quad d_\omega\big([\![x]\!]_c, [\![y]\!]_c\big)$$

for d_ω on Ω turning ω into a final $\overline{F}$-coalgebra

For all coalgebras $c\colon X \to FX$ we want to have:

$$d_c(x,y) \quad = \quad d_\omega\big([\![x]\!]_c, [\![y]\!]_c\big)\,. \tag{1}$$

Especially for $c = \omega$, where ω is an iso and $[\![\cdot]\!]_\omega = \mathrm{id}_\Omega$.

Definition: Behavioral Pseudometric

Define $d_c\colon X \times X \to [0, \top]$ as least fixpoint of:

$$d_c(x,y) = d_c^F\big(c(x), c(y)\big)$$

Theorem: Behavioral Pseudometric

1. $c = \omega$ yields final $\overline{F}$-coalgebra.
2. If $\overline{F}$ preserves isometries, (1) holds for all c.
3. If ω is obtained via the final chain construction, $\overline{F}$ preserves isometries and metrics, then d_ω is a proper metric.

Trace Metrics via Functor Lifting

Coalgebraic Traces in Eilenberg-Moore

Our coalgebras are of the shape

$$c\colon X \to FTX$$

where T is a monad on $\mathbf{Set}$ and $F = B \times \_^A$.

- A nondeterministic automaton is a $\mathbf{Set}$-coalgebra

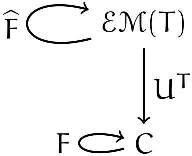
$$c\colon X \to 2 \times (\mathcal{P}_{\mathrm{fin}} X)^A \,.$$

- A probabilistic Moore automaton is a $\mathbf{Set}$-coalgebra

$$c\colon X \to [0,1] \times (\mathcal{D} X)^A \,.$$

Known: To obtain trace semantics via final coalgebra, we need to move to the Eilenberg-Moore category of T.

Eilenberg-Moore & Liftings



Definition: Lifting of a Functor to $\mathcal{EM}(T)$

$\widehat{F}$ is a lifting of F if the diagram commutes.

Proposition: Liftings and Distributive Laws

$$\widehat{F}\colon \mathcal{EM}(T) \to \mathcal{EM}(T) \qquad \xleftrightarrow{\cong} \qquad \lambda\colon TF \Rightarrow FT$$

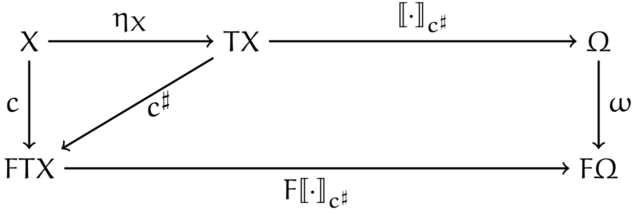
Given λ and an FT-coalgebra $c\colon X \to FTX$ define:

$$c^\sharp := \left(TX \xrightarrow{Tc} TFTX \xrightarrow{\lambda_{TX}} FTTX \xrightarrow{F\mu_X} FTX\right)$$

This F-coalgebra yields a $\widehat{F}$-coalgebra $\mu_X \to \widehat{F}\mu_X$.

Traces in Eilenberg-Moore

via the generalized powerset construction [Silva et al., 2013, Jacobs et al., 2015]



Definition: Trace Equivalence

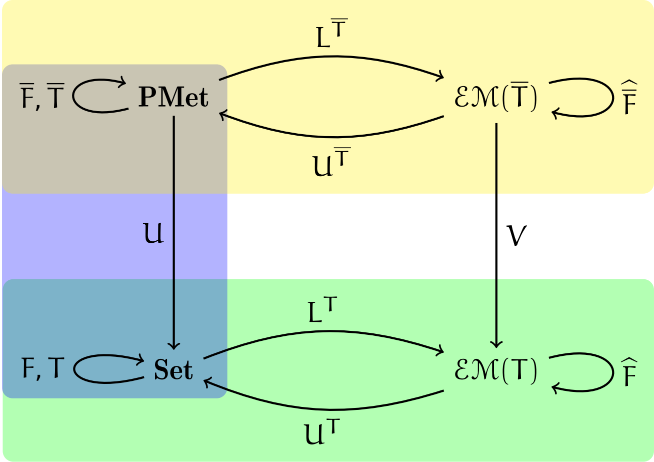
$$\forall x, y \in X : \qquad x \approx_c y \quad \iff \quad [\![\eta(x)]\!]_{c^\sharp} = [\![\eta(y)]\!]_{c^\sharp}$$

Definition: Trace Pseudometric [new! ☺]

$$\forall x, y \in X : \qquad d_c(x, y) \quad = \quad d_\omega\left([\![\eta(x)]\!]_{c^\sharp},\ [\![\eta(y)]\!]_{c^\sharp}\right)$$

for d_ω on Ω turning ω into a final $\overline{F}$-coalgebra

The “Big Picture”



Our Results

... Ensure the Validity of the Aforementioned Approach

- compositionality

$$\overline{F}\,\overline{G} \quad = \quad \overline{FG}$$

- lifting of natural transformations

$$\lambda\colon F \Rightarrow G \quad \rightsquigarrow \quad \overline{\lambda}\colon \overline{F} \Rightarrow \overline{G}$$

- lifting of monads

$$(T, \eta, \mu) \quad \rightsquigarrow \quad (\overline{T}, \overline{\eta}, \overline{\mu})$$

- we obtain the trace pseudometrics of our initial examples!

Final Remarks

Summary, Future and Related Work

Our Contribution

Trace pseudometrics via functor lifting and determinization!

Future/Related Work

- What about coalgebras $X \to TFX$ in $\mathcal{K}\ell(T)$?
- Our lifting approach is clearly inspired by [van Breugel and Worrell, 2005]. What about a logical characterization of distances as in [Desharnais et al., 1999]?
- Directed metrics (simulation distances) [de Alfaro et al., 2009]
- Vary metric on $[0, \top]$ [Chatzikokolakis et al., 2014]
- Algorithms (polynomial-time [Chen et al., 2012], on-the-fly [Bacci et al., 2013],...)
- A fibrational perspective (work in progress)

Bibliography I

Bacci, G., Bacci, G., Larsen, K. G., and Mardare, R. (2013).
On-the-fly exact computation of bisimilarity distances.
In *Proc. of TACAS ’13*, pages 1–15.

Baldan, P., Bonchi, F., Kerstan, H., and König, B. (2014).
Behavioral metrics via functor lifting.
In *FSTTCS 2014*, LIPIcs Series. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

Baldan, P., Bonchi, F., Kerstan, H., and König, B. (2015).
Towards Trace Metrics via Functor Lifting.
In *Proc. of CALCO 2015*, Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

Chatzikokolakis, K., Gebler, D., Palamidessi, C., and Xu, L. (2014).
Generalized bisimulation metrics.
In Baldan, P. and Gorla, D., editors, *CONCUR 2014 – Concurrency Theory*, volume 8704 of *LNCS*, pages 32–46. Springer Berlin Heidelberg.

Chen, D., van Breugel, F., and Worrell, J. (2012).
On the complexity of computing probabilistic bisimilarity.
In *Proc. of FoSSaCS ’12*, volume 7213 of *LNCS*, pages 437–451.

de Alfaro, L., Faella, M., and Stoelinga, M. (2009).
Linear and branching system metrics.
IEEE Transactions on Software Engineering, 25(2).

Desharnais, J., Gupta, V., Jagadeesan, R., and Panangaden, P. (1999).
Metrics for labeled Markov systems.
In *CONCUR’99 Concurrency Theory*, volume 1664 of *LNCS*, pages 258–273. Springer Berlin Heidelberg.

Bibliography II

Jacobs, B., Silva, A., and Sokolova, A. (2015).
Trace semantics via determinization.
Journal of Computer and System Sciences, 81(5):859 – 879.
11th International Workshop on Coalgebraic Methods in Computer Science, CMCS 2012 (Selected Papers).

Silva, A., Bonchi, F., Bonsangue, M. M., and Rutten, J. J. M. M. (2013).
Generalizing determinization from automata to coalgebras.
LMCS, 9(1).

van Breugel, F. and Worrell, J. (2005).
A behavioural pseudometric for probabilistic transition systems.
TCS, 331:115–142.

Villani, C. (2009).
Optimal Transport – Old and New, volume 338 of *A Series of Comprehensive Studies in Mathematics*.
Springer.