# Coalgebraic Behavior Analysis

## From Qualitative to Quantitative Analyses

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Disputation (PhD Defense)

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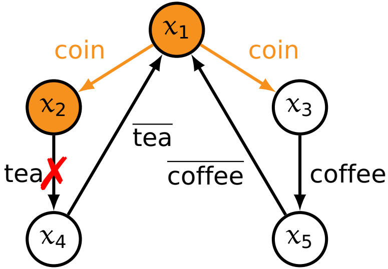
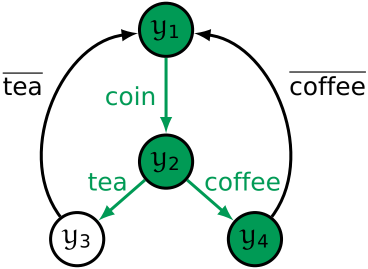
# Introduction and Motivation

# Transition Systems and Behavior

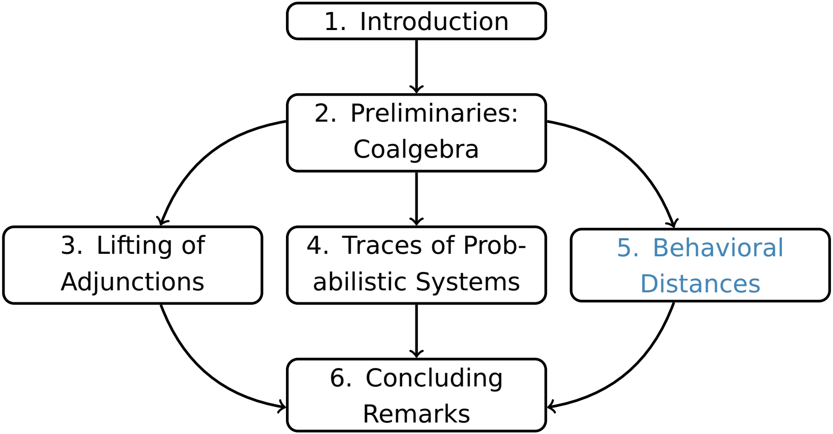
- task: model a tea/coffee vending machine
- allowed sequences: coin, tea, $\overline{\text{tea}}$ and coin, coffee, $\overline{\text{coffee}}$
- machine on the left is wrong (no choice of beverage)
- behavior: traces (sequences) vs. bisimilarity



# My Thesis – Overview

Coalgebraic Behavior Analysis – From Qualitative to Quantitative Analyses



# This Talk – Overview

Main Focus: Behavioral Distances via Functor Lifting (Chapter 5)

# Behavioral Distances via Functor Lifting

# Behavioral Distances

via Functor Lifting (Chapter 5)

- joint work with Paolo Baldan, Filippo Bonchi, Barbara König
- original ideas published at FSTTCS 2014 (New Delhi, India)
- further developments (trace metrics) published at CALCO 2015 (Nijmegen, The Netherlands)
- invited submission to special issue (CALCO'15) of LMCS, submitted for review

Paolo Baldan, Filippo Bonchi, Henning Kerstan, and Barbara König. "Behavioral Metrics via Functor Lifting". In: 34th International Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2014). Ed. by Venkatesh Raman and S. P. Suresh. Vol. 29. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2014, pp. 403–415. doi:10.4230/LIPIcs.FSTTCS.2014.403. arXiv:1410.3385 [cs.LO].

Paolo Baldan, Filippo Bonchi, Henning Kerstan, and Barbara König. "Towards Trace Metrics via Functor Lifting". In: 6th Conference on Algebra and Coalgebra in Computer Science (CALCO'15). Ed. by Lawrence S. Moss and Paweł Sobociński. Vol. 35. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Oct. 2015, pp. 35–49. doi:10.4230/LIPIcs.CALCO.2015.35. arXiv:1505.08105 [cs.LO].

Paolo Baldan, Filippo Bonchi, Henning Kerstan, and Barbara König. "Coalgebraic Behavioral Metrics". In: Logical Methods in Computer Science (2016). – invited submission to special issue, submitted for review –.

# Behavioral Distances

# Behavioral Equivalence vs. Behavioral Distance

## Behavioral Equivalence

Two systems behave exactly the same.

Examples: bisimilarity, trace/language equivalence, failure equivalence ...

## Behavioral Distance

Two systems behave similarly, i.e. they have a small distance.

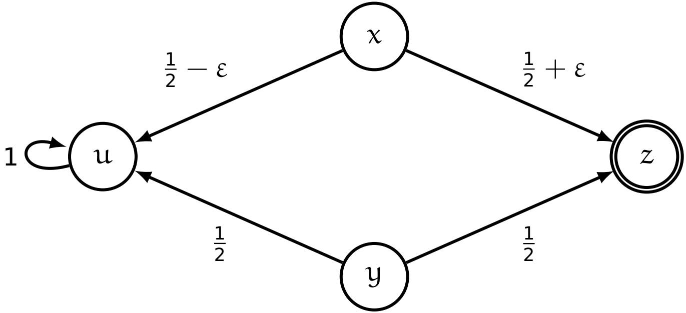
Examples: bisimilarity pseudometric, trace or language pseudometric ...

### Central Contribution

Canonical behavioral pseudometrics via an abstract framework.

# Example: Probabilistic Transition System

Probabilistic Bisimilarity (Larsen and Skou 1989), Example (van Breugel and Worrell 2006)

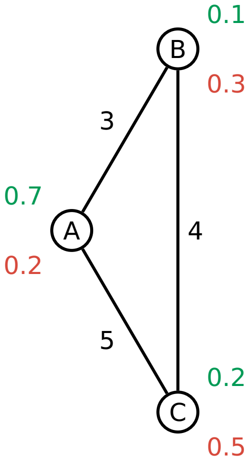



- transitions given by function $\alpha \colon X \to \mathcal{D}_f(X)$
- distance needs to take into account the transitions

# Distances on Probability Distributions

Based on Optimal Transportation (Villani 2009)

Given: $X = \{A, B, C\}$, $d_X : X \times X \to [0, \infty]$, $\mu, \nu : X \to [0, 1]$




### Transportation Problem

- A, B, C: producers/consumers
- supply: $\mu$, demand: $\nu$ (in %)
- transporting one unit costs one monetary unit per unit of distance travelled
- minimize total cost of transport

### Optimal Solution

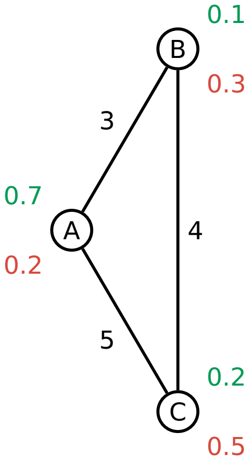
- first use local production (cost 0)
- transport 0.2 from A to B: $3 \cdot 0.2 = 0.6$
- transport 0.3 from A to C: $5 \cdot 0.3 = 1.5$

$$d(\mu, \nu) = 0.6 + 1.5 = 2.1$$

# Distances on Probability Distributions II – Dual View

Based on Optimal Transportation (Villani 2009)

Given: $X = \{A, B, C\}$, $d_X : X \times X \to [0, \infty]$, $\mu, \nu : X \to [0, 1]$




## Transportation Problem (Logistics Firm)

- A, B, C: producers/consumers
- supply: $\mu$, demand: $\nu$ (in %)
- set buying/selling price function pr: $X \to [0, \infty]$ which is competitive, i.e. $|\text{pr}(x) - \text{pr}(y)| \leqslant d_X(x, y)$
- maximize (total) profit

## Optimal Solution

- buy at A for $\text{pr}(A) = 0$
- sell at B for $\text{pr}(B) = 3$
- sell at C for $\text{pr}(C) = 5$
- total: $0 + 3 \cdot 0.2 + 5 \cdot 0.3 = 2.1$

# Lifting Functors

**Definition: Functor**

A functor $F\colon \mathcal{S}et \to \mathcal{S}et$ maps

- every set $X$ to a set $FX = F(X)$, and
- every function $f\colon X \to Y$ to a function $Ff\colon FX \to FY$

such that

- identities are preserved:

$$F\left(X \xrightarrow{\mathrm{id}_X} X\right) = FX \xrightarrow{\mathrm{id}_{FX}} FX$$

- composition is preserved:

$$F\left(X \xrightarrow{f} Y \xrightarrow{g} Z\right) = FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ$$

# Examples of Functors

| Functor $F$ | Name | Value $FX$ on a set $X$ |
|---|---|---|
| $\mathcal{P}_f$ | finite powerset | $\{S \subseteq X \mid S \text{ finite}\}$ |
| $\mathcal{D}_f$ | finitely supported distributions | $\left\{ X \xrightarrow{P} [0,1] \;\middle|\; \begin{matrix} \text{supp}(P) < \infty \\ \sum_{x \in X} P(x) = 1 \end{matrix} \right\}$ |
| $A \times \_$ | product with $A$ | $A \times X$ |
| $\_ + B$ | coproduct with $B$ (disjoint union) | $X + B \cong \left\{ (x,1), (b,2) \;\middle|\; \begin{matrix} x \in X \\ b \in B \end{matrix} \right\}$ |
| $F_2 \circ F_1$ | composition of functors $F_1, F_2$ | $F_2(F_1 X)$ |

# Pseudometric Spaces & Nonexpansive Functions

**Definition: Pseudometric**

Let $\top \in (0, \infty]$ be fixed. Given a set $X$, a pseudometric on $X$ is a function $d \colon X \times X \to [0, \top]$ which satisfies, for all $x, y, z \in X$,

1. $d(x, x) = 0$, (reflexivity)
2. $d(x, y) = d(y, x)$, and (symmetry)
3. $d(x, z) \leqslant d(x, y) + d(y, z)$. (triangle inequality)

It is called a metric if it additionally satisfies, for all $x, y \in X$,

4. $d(x, y) = 0 \implies x = y$.

The pair $(X, d)$ is called (pseudo)metric space.

**Definition: Nonexpansive Function**

For pseudometric spaces $(X, d_X)$, $(Y, d_Y)$ a function $f \colon X \to Y$ is called nonexpansive if $d_Y \circ (f \times f) \leqslant d_X$.

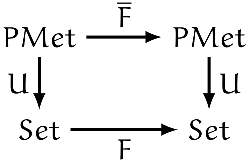
# Lifting of a Functor

**Definition: Functor on $\mathcal{PMet}$**

Same as for $\mathcal{Set}$, replace sets by pseudometric spaces and functions by nonexpansive functions.

**Definition: Lifting of a Functor**

Let $U \colon \mathcal{PMet} \to \mathcal{Set}$ be the forgetful functor. A lifting of $F \colon \mathcal{Set} \to \mathcal{Set}$ is a functor $\overline{F}$ on $\mathcal{PMet}$ such that $U\overline{F} = FU$.




## In Plain Terms: Our Task

Given: functor $F$ on $\mathcal{Set}$ and a pseudometric space $(X, d_X)$
Define: pseudometric $d_X^F$ on $FX$

# Transportation Plans Revisited

Task: Given $F$, pseudometr. space $(X, d_X)$ – Define Pseudometric $d_X^F$ on $FX$.

$$\mu, \nu \in \mathcal{D}_f(X), \quad t \in \mathcal{D}_f(X \times X), \quad t \colon \underbrace{X}_{\text{from}} \times \underbrace{X}_{\text{to}} \to \underbrace{[0,1]}_{\text{percentage}}$$

supply is used up

$$\forall x \in X. \sum_{y \in X} t(x,y) = \mu(x)$$

$$\iff \quad \mathcal{D}_f \pi_1(t) = \mu$$

demand is satisfied

$$\forall y \in X. \sum_{x \in X} t(x,y) = \nu(y)$$

$$\iff \quad \mathcal{D}_f \pi_2(t) = \nu$$

cost of transportation plan depends on $d_X$

$$\text{cost}(d_X)(t) = \sum_{(x,y) \in X^2} t(x,y) \cdot d_X(x,y)$$

# Towards The Generalized Wasserstein Lifting

Task: Given $F$, pseudometr. space $(X, d_X)$ – Define Pseudometric ${d_X}^{\downarrow F}$ on $FX$.

- special functor $F = \mathcal{D}_f$, $\mu, \nu \in \mathcal{D}_f X$

$$ {d_X}^{\downarrow \mathcal{D}_f}(\mu, \nu) = \min \left\{ \mathrm{cost}(d_X)(t) \;\middle|\; \begin{array}{c} t \in \mathcal{D}_f(X \times X) \\ \mathcal{D}_f \pi_1(t) = \mu \\ \mathcal{D}_f \pi_2(t) = \nu \end{array} \right\} $$

- general functor $F\colon \mathit{Set} \to \mathit{Set}$, $t_1, t_2 \in FX$

$$ {d_X}^{\downarrow F}(t_1, t_2) = \inf \left\{ \widetilde{F}(d_X)(t) \;\middle|\; \begin{array}{c} t \in F(X \times X) \\ F\pi_1(t) = t_1 \\ F\pi_2(t) = t_2 \end{array} \right\} $$

where $\widetilde{F}\colon \mathit{Set}/[0,\top] \to \mathit{Set}/[0,\top]$ is the evaluation functor

$$ \widetilde{F}\left(Y \xrightarrow{f} [0,\top]\right) = FY \xrightarrow{Ff} \underbrace{F[0,\top] \xrightarrow{ev_F} [0,\top]}_{\text{evaluation function}} $$

# Putting It Together: The Wasserstein Lifting

Task: Given $F$, pseudometr. space $(X, d_X)$ – Define Pseudometric $d_X{}^{\downarrow F}$ on $FX$.

**Definition: Wasserstein Function**

For $t_1, t_2 \in FX$ define

$$d_X{}^{\downarrow F}(t_1, t_2) = \inf \left\{ \widetilde{F}(d_X)(t) \;\middle|\; \begin{array}{c} t \in F(X \times X) \\ F\pi_1(t) = t_1 \\ F\pi_2(t) = t_2 \end{array} \right\}$$

**Theorem: Wasserstein Pseudometric**

$d^{\downarrow F}$ is a pseudometric if

- $F$ preserves weak pullbacks
- $ev_F$ satisfies some constraints (well-behaved)

# Duality? – A Glimpse at the Kantorovich Lifting

Task: Given $F$, pseudometr. space $(X, d_X)$ – Define Pseudometric $d_X{}^F$ on $FX$.

- Wasserstein Lifting: For $t_1, t_2 \in FX$ we define

$$d_X{}^{\downarrow F}(t_1, t_2) = \inf \left\{ \widetilde{F}(d_X)(t) \;\middle|\; \begin{array}{c} t \in F(X \times X) \\ F\pi_1(t) = t_1 \\ F\pi_2(t) = t_2 \end{array} \right\}$$

- Kantorovich Lifting: For $t_1, t_2 \in FX$ we define

$$d_X{}^{\uparrow F}(t_1, t_2) = \sup \left\{ d_e\left(\widetilde{F}f(t_1), \widetilde{F}f(t_2)\right) \;\middle|\; \begin{array}{c} f\colon X \to [0, \top] \\ d_e \circ (f \times f) \leqslant d_X \end{array} \right\}$$

- Kantorovich-Rubinstein-Duality:

$$d_X{}^{\uparrow F}(t_1, t_2) = d_X{}^{\downarrow F}(t_1, t_2)$$

# Evaluation Functions and Resulting Pseudometrics

| $F$ | $\top$ | evaluation | resulting metric (KR-duality!) |
|---|---|---|---|
| $\mathcal{P}_f$ | $\infty$ | $ev(S) = \max S$ | Hausdorff |
| $\mathcal{D}_f$ | 1 | $ev(P) = \mathbb{E}_P[\mathrm{id}_X]$ | Kantorovich / Wasserstein |
| $\_ + \_$ | * | $ev((x,i)) = x$ | $d\big((x,i),(y,j)\big) = \begin{cases} d_i(x,y), & i = j \\ \top, & i \neq j \end{cases}$ |
| $\_ \times \_$ | * | $ev(x,y) = \max(x,y)$ | $d_\infty\big((x_1,y_1),(x_2,y_2)\big) = \max\big(d_X(x_1,x_2), d_Y(y_1,y_2)\big)$ |
| $\_ \times \_$ | * | $ev(x,y) \mapsto x + y$ | $d_+\big((x_1,y_1),(x_2,y_2)\big) = d_X(x_1,x_2) + d_Y(y_1,y_2)$ |

* = arbitrary $\top \in (0,\infty]$

# Behavioral Distances via Functor Lifting

# Transition Systems are Coalgebras

Explored in *Universal Coalgebra: A Theory of Systems* (Rutten 2000)

**Definition: Coalgebra**

Let $F$ be a functor on $\mathcal{S}et$. An $F$-coalgebra is a function $\alpha\colon X \to FX$. The carrier $X$ is the state space and the function describes the transitions of the respective transition system.

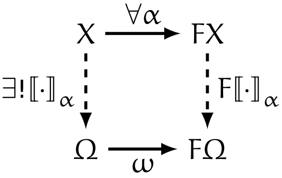
| coalgebra $\alpha$ | transition system |
|---|---|
| $X \to \mathcal{P}_f X$ | non-deterministic branching |
| $X \to \mathcal{D}_f X$ | probabilistic branching |
| $X \to A \times X$ | labelling |
| $X \to X + \{\checkmark\}$ | termination, exceptions, failure |
| $X \to \mathcal{P}_f(A \times X + \{\checkmark\})$ | NFA |

# Final Coalgebra and Behavior

The Essential Basics in a Nutshell (i.e. on one Slide)

**Definition: Final Coalgebra**

$\omega\colon \Omega \to F\Omega$ is final, if for all $\alpha\colon X \to FX$ there is a unique coalgebra homomorphism $[\![\cdot]\!]_\alpha\colon X \to \Omega$.




**Definition: Behavioral Equivalence (Well-Known)**

$$\forall x, y \in X: \qquad x \sim_\alpha y \quad \Longleftrightarrow \quad [\![x]\!]_\alpha = [\![y]\!]_\alpha$$

**Definition: Bisimilarity Pseudometric (New!)**

$$\forall x, y \in X: \qquad bd_\alpha(x, y) \quad = \quad d_\Omega\big([\![x]\!]_\alpha, [\![y]\!]_\alpha\big)$$

for suitable $d_\Omega$ on $\Omega$ which turns $\omega$ into a final $\overline{F}$-coalgebra

# The Bisimilarity Pseudometric

For all $\alpha\colon X \to FX$, we would like to have:

$$\forall x, y \in X \quad bd_\alpha(x, y) = bd_\alpha^F\big(\alpha(x), \alpha(y)\big) \tag{1}$$

Definition: Final Coalgebra Pseudometric

Define $d_\Omega\colon \Omega \times \Omega \to [0, \top]$ as least fixpoint of:

$$\forall x, y \in \Omega \quad d(x, y) = d^F\big(\omega(x), \omega(y)\big)$$

Theorem: Bisimilarity Pseudometric

1. With $d_\Omega$ we obtain the final $\overline{F}$-coalgebra
2. If $\overline{F}$ preserves isometries, equation (1) holds for all $\alpha$.
3. If $\omega$ is obtained via the final chain construction, $\overline{F}$ preserves isometries and metrics, then $d_\Omega$ is a proper metric.

# Final Remarks

# Summary and Related Work

## Central Contribution

A framework for behavioral pseudometrics for systems that are coalgebras for a $\mathcal{S}et$-functor based on the notion of functor lifting.

- ✓ Linear distances (trace pseudometrics): Chapter 5 and the CALCO'15 paper (Baldan, Bonchi, Kerstan, and König 2015)
- ▶ Logical characterization of distances for probabilistic systems (Desharnais, Edalat, and Panangaden 1998)
- ▶ Directed metrics for metric transition systems (de Alfaro, Faella, and Stoelinga 2009)
- ▶ Generalized bisimulation metrics (Kantorovich based) for probabilistic systems $[0, \top]$ (Chatzikokolakis, Gebler, Palamidessi, and Xu 2014)
- ▶ Algorithms for probabilistic systems: polynomial-time (Chen, van Breugel, and Worrell 2012), on-the-fly (Giorgio Bacci, Giovanni Bacci, Larsen, and Mardare 2013)

# Further Contributions

# Lifting Adjunctions

## to Coalgebras to (Re)Discover Automata Constructions (Chapter 3)

- joint work with Barbara König and Bram Westerbaan
- main result: lifting an adjunction to coalgebras yields many well-known constructions (e.g. determinization, co-determinization) on automata
- very short version presented at CALCO Early Ideas 2013 in Warsaw, Poland received best presentation award but paper was not accepted to appear in proceedings
- full paper accepted at CMCS 2014 (Grenoble, France)

Henning Kerstan, Barbara König, and Bram Westerbaan. "Lifting Adjunctions to Coalgebras to (Re)Discover Automata Constructions". In: Coalgebraic Methods in Computer Science. Ed. by Marcello M. Bonsangue. Vol. 8446. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2014, pp. 168–188. doi:10.1007/978-3-662-44124-4_10.

# Trace Semantics

for Continuous Probabilistic Transition Systems (Chapter 4)

- joint work with Barbara König, based on my diploma thesis
- main result: finite and infinite traces for continuous probabilistic transition systems (aka labelled Markov processes) arise as final coalgebra of suitable functors in Kleisli categories
- published at CONCUR 2012 (Newcastle, UK)
- revised and extended version in a special issue of LMCS


Henning Kerstan. "Trace Semantics for Probabilistic Transition Systems – A Coalgebraic Approach". Diplomarbeit. Universität Duisburg-Essen, Sept. 2011.

Henning Kerstan and Barbara König. "Coalgebraic Trace Semantics for Probabilistic Transition Systems Based on Measure Theory". In: CONCUR 2012 – Concurrency Theory. Ed. by Maciej Koutny and Irek Ulidowski. Vol. 7454. Lecture Notes in Computer Science. Springer Berlin Heidelberg, Sept. 2012, pp. 410–424. doi:10.1007/978-3-642-32940-1_29.

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# Concluding Remarks

# From Qualitative To Quantitative Analyses

## Main Contributions in a Nutshell

1. Adjunction Lifting (Generic)
   - before: abstract (2-categorical) theory of adjunction lifting
   - now: applicability of the theory; determinization constructions

2. Traces for Probabilistic Systems (Quantitative)
   - before: focus on finite/countable systems and finite traces
   - now: continuous systems (uncountable state spaces) and infinite traces

3. Behavioral Distances (Quantitative)
   - before: focus on equivalences (except for specific cases)
   - now: rich, general theory for pseudometrics

4. A new introduction to coalgebraic modelling and behavior analysis (Introduction/Preliminaries)

# My Publications

Preprint of my Thesis: `http://henningkerstan.org/thesis`


- Paolo Baldan, Filippo Bonchi, Henning Kerstan, and Barbara König. "Coalgebraic Behavioral Metrics". In: Logical Methods in Computer Science (2016). – invited submission to special issue, submitted for review –.

- Paolo Baldan, Filippo Bonchi, Henning Kerstan, and Barbara König. "Towards Trace Metrics via Functor Lifting". In: 6th Conference on Algebra and Coalgebra in Computer Science (CALCO'15). Ed. by Lawrence S. Moss and Paweł Sobociński. Vol. 35. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Oct. 2015, pp. 35–49. doi:10.4230/LIPIcs.CALCO.2015.35. arXiv:1505.08105 [cs.LO].

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Cédric Villani. Optimal Transport – Old and New. Vol. 338. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2009. doi:10.1007/978-3-540-71050-9.